Physics S1 (2025-2026)

Part 2: Oscillating systems

I. The harmonic oscillator model

The harmonic oscillator is a model describing an ideal oscillator where friction is neglected and, consequently, whose amplitude remains constant over time. This model is used in many fields of physics: classical mechanics, electricity, electronics, optics, acoustics, quantum mechanics, etc.

1. Free-body diagram

In this section of the course, we will consider the following oscillator, consisting of a mass m at a point M attached to a spring of length l_0 when unstretched, and subjected to the following three forces:

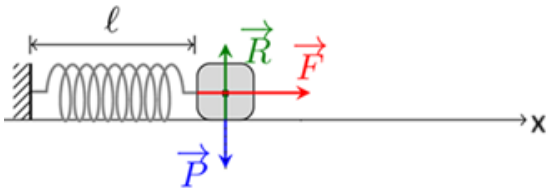


Figure 1: Free-body diagram for a spring. (Source)

- its weight $\vec{P}$, a vertical force pointing downwards,
- the surface reaction $\vec{R}$, a vertical force pointing upwards, and
- the restoring force from the spring $\vec{F}_{spring}$, a horizontal force directed either to the left or to the right depending on the elongation of the spring.

The restoring force is a force proportional to the elongation Δl of the spring and to a constant k that characterizes its stiffness and is expressed in N/m. Its expression is as follows:

$$\vec{F}_{spring} = -\,k\,\Delta l\,\vec{u}_x$$

where:

- $\Delta l = l - l_0$
- The quantity l_0 corresponds to the length of the spring at rest.
- The quantity l corresponds to the length of the spring at time t.

Let's now discuss the direction of the restoring force. We place the origin of the x-axis at the point where the spring is in its rest position, i.e. where the elongation is zero. When the spring is out of equilibrium, two cases are possible:

- If $\Delta l < 0$, the spring is compressed, and the force is directed in the same direction as the $\vec{u}_x$ vector.
- If $\Delta l > 0$, the spring is extended, and the force is directed in the direction opposite to the $\vec{u}_x$ vector.

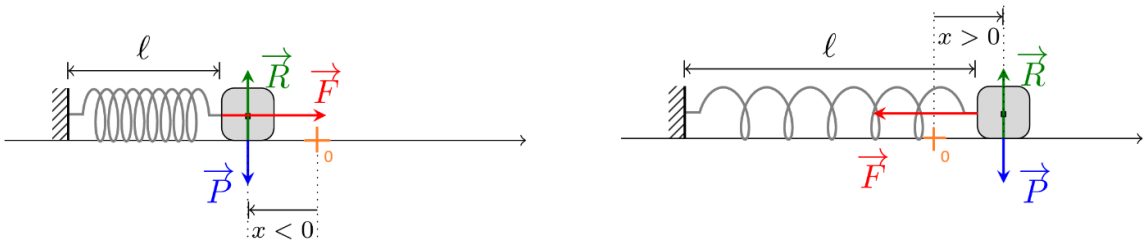


Figure 2: Free-body diagram in the case of a compressed (left) and an extended spring (right). (Source)

Given that the origin of the x-axis is at the spring's point of equilibrium, as shown in the figures above, we have that:

- At rest: $\Delta l = l - l_0 = 0$
- In all cases: $\Delta l = x$

When the spring extension is positive (Fig.2 on the right), then the restoring force is negative and directed to the left. Similarly, if the spring extension is negative (Fig.2 on the left), then the restoring force is positive and directed to the right.

2. Application of Newton's second law

To describe the motion of the mass m, we apply Newton's second law:

$$\sum_i \vec{F}_i = m\vec{a}$$

$$\vec{P} + \vec{R} + \vec{F}_{spring} = m\vec{a}$$

By projecting on the x-axis, we obtain

$$0 + 0 - kx = m\ddot{x}$$

which can be written as

$$\ddot{x} + \frac{k}{m}x = 0$$

This is a second-order differential equation, that is generally written in the following form:

$$\ddot{x} + \omega_0^2 x = 0$$

The quantity ω_0 is known as the natural frequency (or eigenfrequency) of the system:

$$\omega_0 = \sqrt{\frac{k}{m}}$$

3. Solution of the differential equation

We must now find the function $x(t)$ that satisfies the differential equation. The solution is not unique: we need to use the initial conditions to find the correct one. These initial conditions include

- the initial position $x(0) = x_0$ (measured in meters), and
- the initial velocity $v(0) = v_0$ (measured in meters per second).

It can be shown that the general solution of the differential equation can be written in the following form:

$$x(t) = A\cos(\omega_0 t) + B\,sin(\omega_0 t)$$

Case 1: The spring is stretched to a position x_0 and is released with zero initial velocity $v_0 = 0$

Using the initial condition for the position we obtain

$$x(0) = A\cos(0) + B\,sin(0) = A = x_0$$

Thus, we can write

$$x(t) = x_0\cos(\omega_0 t) + B\,sin(\omega_0 t)$$

By differentiating this expression, we get

$$\dot{x}(t) = -x_0\,\omega_0\sin(\omega_0 t) + B\,\omega_0\,cos\,(\omega_0 t)$$

Using now the initial condition for the velocity, we obtain

$$\dot{x}(t) = -x_0\,\omega_0\sin(0) + B\,\omega_0\cos(0) = B = 0$$

Therefore, the velocity is

$$\dot{x}(t) = -x_0\,\omega_0\sin(\omega_0 t)$$

If we differentiate this expression once more, we get

$$\ddot{x}(t) = -\,x_0\,\omega_0^2\cos(\omega_0 t) = -\,x(t)\,\omega_0^2$$

which is our differential equation. The spring oscillates between $-\,x_0$ and x_0, as expected for a model without friction.

Case 2: The spring is released with an initial velocity v_0 from the point of equilibrium $x_0 = 0$

Using the initial condition for the position, we now obtain

$$x(0) = A\cos(0) + B\sin(0) = A = 0$$

That is,

$$x(t) = B\sin(\omega_0 t)$$

By differentiating this expression, we get

$$\dot{x}(t) = B\,\omega_0\cos(\omega_0 t)$$

Using the initial condition for the velocity,

$$\dot{x}(t) = B\,\omega_0\cos(0) = B\omega_0 = \ v_0 \qquad \Longrightarrow \qquad B = \frac{v_0}{\omega_0}$$

Therefore,

$$\dot{x}(t) = v_0\ cos\ (\omega_0 t)$$

If we differentiate once more,

$$\ddot{x}(t) = -\,v_0\,\omega_0\sin(\omega_0 t) = -\,x(t)\,\omega_0^2$$

We obtain our differential equation. Now the spring oscillates between $-\frac{v_0}{\omega_0}$ and $\frac{v_0}{\omega_0}$.

It can be easily verified that the sum of the two solutions found above is the solution of the differential equation when the initial conditions are (x_0, v_0):

$$x(t) = x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\,sin(\omega_0 t)$$

Definition 10: Solution of the harmonic oscillator

The generic solution of the harmonic oscillator is

$$x(t) = A\cos(\omega_0 t) + \ B\sin(\omega_0 t)$$

where A and B are constants to be determined by the initial conditions. This solution can also be written in the form

$$x(t) = A\cos(\omega_0 t + \phi)$$

where A and ϕ are, again, constants to be determined by the initial conditions. In this case, A is the amplitude of the motion (measured in meters) and ϕ the initial phase (measured in radian).

4. Solution curve

The oscillations of point M oscillates are sinusoidal with amplitude x_0 and natural period

$$T_0 = \frac{2\pi}{\omega_0} = \ 2\pi\sqrt{\frac{m}{k}}$$

The oscillator is called harmonic because its oscillations have a constant amplitude and a constant natural period, the value of which depends only on the characteristics of the spring-mass system.

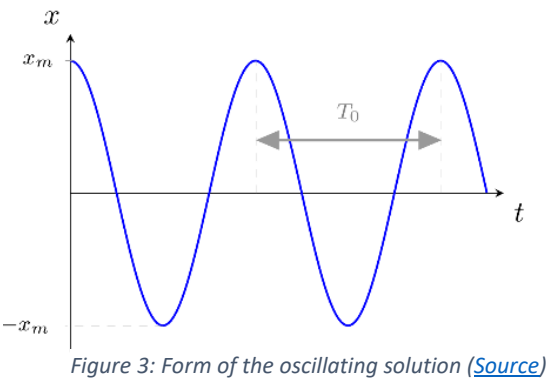


Figure 3: Form of the oscillating solution (Source)

5. Energy of the harmonic oscillator

The kinetic energy of the harmonic oscillator is given by the well-known formula

$$E_k = \frac{1}{2}mv^2$$

Because of the presence of the spring, the oscillator has also potential elastic energy, which is given by the expression

$$E_p = \frac{1}{2}kx^2$$

Using the solution for the harmonic oscillator, we can rewrite these energies as follows:

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}\,mA^2\omega_0^2\sin^2(\omega_0 t + \phi)$$

$$E_p = \frac{1}{2}kx^2 = \frac{1}{2}\,kA^2\cos^2(\omega_0 t + \phi)$$

Using that $\omega_0 = \sqrt{\frac{k}{m}}$, the kinetic energy can be rewritten as

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}\,kA^2\sin^2(\omega_0 t + \phi)$$

We can now calculate the mechanical energy:

$$E_M = \frac{1}{2}\,kA^2\sin^2(\omega_0 t + \phi) + \frac{1}{2}\,kA^2\cos^2(\omega_0 t + \phi)$$

$$E_M = \frac{1}{2}\,kA^2(\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi))$$

$$E_M = \frac{1}{2}\,kA^2$$

We see that the total energy of the harmonic oscillator is constant in time and is proportional to the square of the amplitude.

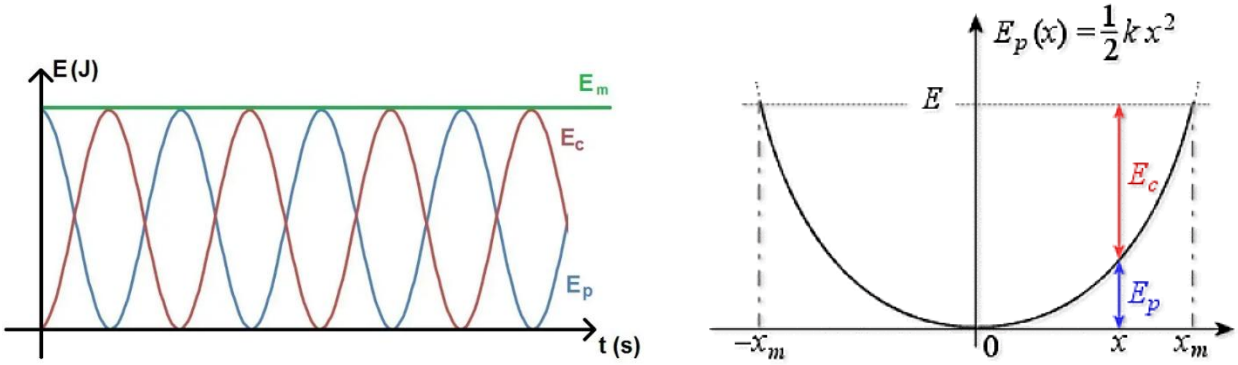


Figure 4: Kinetic, potential and mechanical energy of a harmonic oscillator as a function of time on the left (Source), and the parabolic form of the potential energy on the right (Source).

II. The damped oscillator model

1. Model and differential equation

We now assume that the mass attached to the spring is subjected, in addition to the forces mentioned before, to a force of drag (e.g. air resistance) given by the expression

$$\vec{f} = -\alpha\vec{v}$$

Using Newton's second law and projecting on the x-axis, we get:

$$\vec{P} + \vec{R} + \vec{F}_{spring} + \vec{f} = m\vec{a}$$

$$0 + 0 - kx - \alpha\dot{x} = m\ddot{x}$$

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0$$

As before, this is a second-order differential equation. It is often written as

$$\ddot{x} + \frac{1}{\tau}\dot{x} + \omega_0^2 x = 0$$

where τ is a characteristic timescale of the system defined as $\tau = \frac{m}{\alpha}$. Another common formulation of this equation is the following

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0$$

where $\lambda = \frac{1}{2\tau}$ describes the damping exerted on the system.

Solving this equation is beyond the scope of this course; here we will only discuss the solutions, for which we need to distinguish 3 different cases, depending on the values of the parameters λ and ω_0:

- If $\lambda < \omega_0$, damping is weak: the solution is under-damped.
- If $\lambda > \omega_0$, damping is strong: the solution is over-damped.
- If $\lambda = \omega_0$, damping is critical: the solution is critically damped.

(Note: in French, the terminology used for these three cases is different: they are called "régime pseudo-périodique" for $\lambda < \omega_0$, "régime apériodique" for $\lambda > \omega_0$, and "régime apériodique critique" for $\lambda = \omega_0$.)

2. The under-damped case

If $\lambda < \omega_0$, the solution has the form

$$x(t) = e^{-\lambda t}\big(A\cos(\omega t) + B\sin(\omega t)\big)$$

There is strong similarity with the periodic case of the harmonic oscillator, but there is also an exponential factor in front of the functions of sine and cosine. This factor results in a continuous drop of the amplitude of the $x(t)$ function.

The parameter ω is different than the natural frequency ω_0, and is called the pseudo-pulsation.

3. The over-damped case

If $\lambda > \omega_0$, the solution takes the following form:

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

where the parameters r_1 et r_2 are negative. We note that there are no longer any periodic terms: the solution is simply two decaying exponentials.

4. The critically damped case

In the special case where $\lambda = \omega_0$, the solution takes the form

$$x(t) = (A + Bt)e^{-\omega_0 t}$$

We note that there is no periodic term here either. With this solution, the return to equilibrium is the fastest.

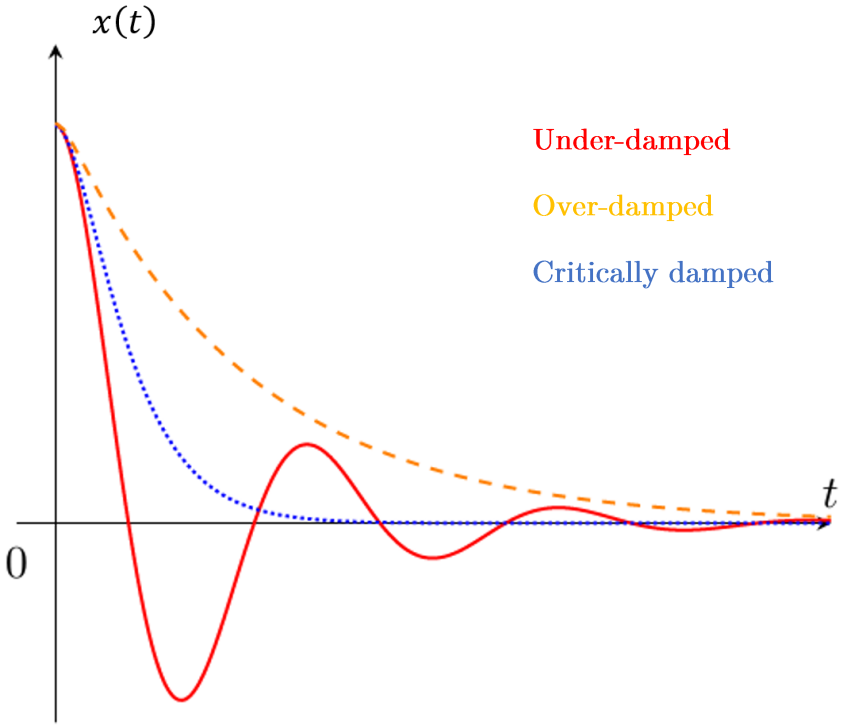


Figure 5 : Solutions of the damped oscillator model in the under-damped, over-damped and critically damped cases. (Source)

III. Waves and signals

1. Signals

A signal is information that depends on time or space. For example, when measuring a physical quantity, researchers measure a signal that they then analyze. Nowadays, these signals are often electric signals, e.g. phone accelerometers, microphones, GPS, CCD sensors in cameras, etc.

In this course, we will limit ourselves to periodic signals, that is to signals that are repeated identically over time. A signal can be expressed mathematically as a function of time and spatial variables. To describe a periodic function, we use the following physical quantities:

- The amplitude A: the maximum value of the signal
- The period T: the time needed for a repetition of the signal
- The frequency f:

$$f = \frac{1}{T}$$

- The velocity c at which the signal is propagates
- The wavelength λ:

$$\lambda = \frac{c}{f}$$

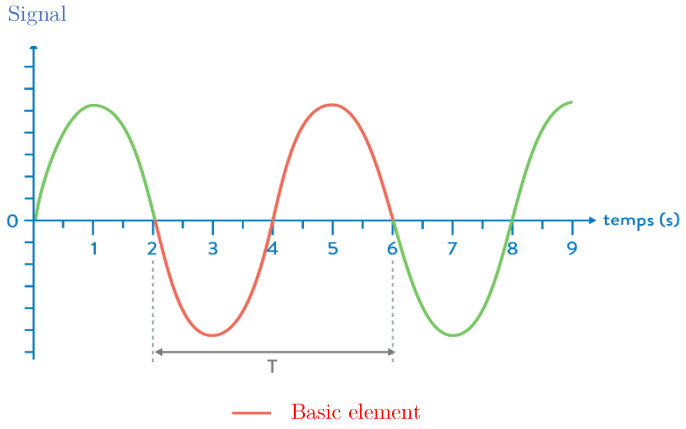


Figure 6 : Time dependence of a periodic signal.

The following table shows some examples of signals.

Type of signal	Oscillating physical quantity	Frequency range
Electric	Current, voltage	50-60 Hz
Acoustic	Pressure	20 Hz – 20 kHz (human ear), infrasounds (<20Hz), ultrasounds (>20kHz)
Electromagnetic	Electric and magnetic fields	kHz-MHz (for radio waves) to 10^{18}Hz and above (for γ rays)

2. Travelling Waves

Definition 11 : Travelling wave

A wave is a periodic signal in which a physical quantity oscillates in time and space. A travelling wave propagates with a velocity c without any displacement of matter.

To describe a travelling wave mathematically, two common simplifying assumptions are that the wave is

- **a plane wave**, i.e. it propagates in only one dimension.
- **a monochromatic wave**, i.e. it is perfectly sinusoidal.

A travelling plane wave propagating with velocity c along the x-axis in the direction of increasing x values can be written as:

$$s(x,t) = f(t - \frac{x}{c})$$

Similarly, a travelling plane wave propagating with velocity c along the x-axis in the direction of decreasing x values is given by

$$s(x,t) = f(t + \frac{x}{c})$$

Definition 12: Travelling plane monochromatic waves

A travelling plane monochromatic wave propagating along the x-axis in the direction of increasing x-values with velocity c can be written as:

$$s(x,t) = A\cos(\omega t - kx + \phi)$$

Where A is the amplitude of the wave (its maximum value), $\omega = 2\pi f$ is the angular frequency, $k = \frac{\omega}{c}$ is the wavenumber, and ϕ is the initial phase (that is determined by the initial conditions).

3. Phase shift

When comparing two signals of travelling plane monochromatic waves with the same frequency, the phase difference or phase shift $\Delta\phi$ of signal $s_2(t)$ relative to signal $s_1(t)$ is given by

$$\Delta\phi = \phi_2 - \phi_1 = -\omega\Delta t$$

where Δt is the time delay of signal $s_2(t)$ relative to signal $s_1(t)$.

- If $\Delta\phi > 0$, then signal $s_2(t)$ is ahead of phase with respect to signal $s_1(t)$.
- If $\Delta\phi < 0$, then signal $s_2(t)$ is phase-delayed relative to signal $s_1(t)$.

The following specific cases are also of interest:

- If $\Delta\phi = 0$, the signals are in phase.
- If $\Delta\phi \neq 0$, the signals are out of phase.
- If $\Delta\phi = \pi$, the signals have opposite phases
- If $\Delta\phi = \frac{\pi}{2}$, the signals are in quadrature.

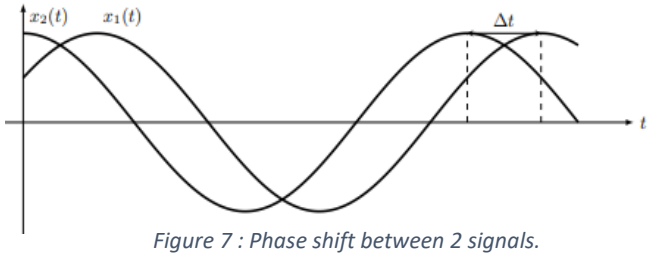


Figure 7 : Phase shift between 2 signals.

4. Standing Waves

Waves can be constrained in various ways, as is the case with a guitar string that is attached at both ends. The attachment points prevent the ends of the string from oscillating. In mathematical modeling, we refer to these as boundary conditions.



Figure 8: Standing waves on a guitar string.

This is the case with most musical instruments: in a flute, whether the tube is closed or open at both ends will impose certain conditions, particularly on the air pressure at these points.

In these cases, we have waves that do not propagate in space and are therefore called **standing waves**. A standing wave is characterized by nodes of vibration, which are points in space where the amplitude of the wave is zero, and antinodes of vibration, which are points in space where the amplitude of the wave is maximum, as shown in Figure 9.

To explain mathematically these characteristics, let's consider a guitar string of length L with bound ends and a plane monochromatic wave propagating on it to the right, given by

$$s_1(x,t) = A\cos(\omega t - kx)$$

When this wave reaches the bound end of the string, it will be reflected. The reflected wave will have the same amplitude, initial phase and frequency, but it will propagate in the opposite direction, i.e. it can be written as:

$$s_2(x,t) = A\cos(\omega t + kx)$$

The total signal is then the sum of these two waves:

$$s(x,t) = A\cos(\omega t - kx) + A\cos(\omega t + kx)$$

$$s(x,t) = 2A\sin(\omega t)\sin(kx)$$

Notice that the time and space coordinates are decoupled.

Since the end of the string is fixed, $s(L,t) = 0$ at all times t. This means that

$$\sin(kL) = 0 \Rightarrow k_n L = n\pi \quad \text{with } n \in \mathbb{N}^*$$

We thus find that the waves on the string must satisfy the following condition

$$k_n = \frac{n\pi}{L} \Leftrightarrow \omega_n = \frac{n\pi c}{L} \quad \text{with } n \in \mathbb{N}^*$$

Or equivalently for the wavelength, given that $\lambda = \frac{c}{f} = \frac{2\pi c}{\omega} = \frac{2\pi}{k}$,

$$\lambda_n = \frac{2L}{n} \qquad \text{with } n \in \mathbb{N}^*$$

These are known as vibrational modes. A vibrational mode is a harmonic standing wave of the form

$$s_n(x,t) = 2A \sin\left(\frac{n\pi x}{L}\right) \sin(\omega t)$$

All points on the string oscillate sinusoidally with the same frequency. The first mode s_1 is called the fundamental mode, and the following modes are called harmonics. Figure 9 shows the first 4 harmonics of a string.

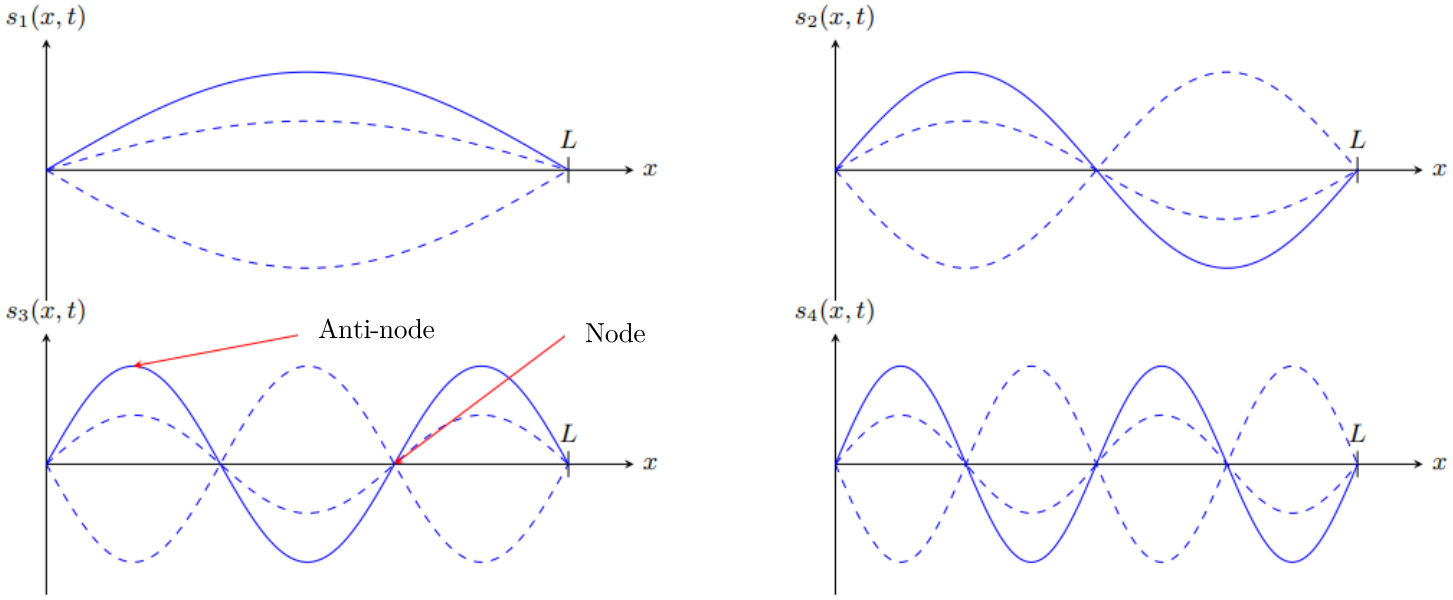


Figure 9: Fundamental and first harmonics of a string with fixed ends.